

Closed Form Evaluation of Symmetric Two-Sided Complex Integrals

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Evaluation of two-sided complex integrals of the form

$$I_n = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dS G(S)G(-S)$$

and

$$I_n = \frac{1}{2\pi i} \oint_{\text{Unit Circle}} \frac{dZ}{Z} G(Z)G(Z^{-1})$$

is often required when analyzing linear systems to determine signal variances resulting from stochastic inputs and system noise bandwidths. Presented are algebraic solutions of both the above integrals in a closed matrix equation form using coefficients of the numerator and denominator polynomials of the function G .

I. Introduction

In the analysis of linear systems for output and internal signal variance caused by noisy input signals, and for the analysis of noise bandwidths of such systems, the following integrals often require evaluation.

$$I_n = \frac{1}{2\pi i} \oint_{\text{Unit Circle}} \frac{dZ}{Z} G(Z)G(Z^{-1}) \quad (1)$$

$$I_n = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dS G(S)G(-S) \quad (2)$$

These integrals are two-sided integrals in the complex plane. Integral (2) arises in the analysis of continuous systems where S is the Laplace transform complex frequency. $G(S)$ is the ratio of two polynomials in S , with the denominator polynomial of degree n being at least 1 degree higher than the numerator polynomial. The zeros of the denominator polynomial are known as the poles of $G(S)$ and are assumed to be located in the left side of the complex S -plane.

Integral (1) arises in the analysis of sampled data systems where Z is equal to e^{ST} , T is the sampling time, and e is the base of the natural logarithm. $G(Z)$ is the ratio of two polynomials in Z , with the denominator polynomial of degree n being equal to, or greater than, the degree of the numerator polyno-

mial. The poles of $G(Z)$ are assumed to lie within the unit circle of the Z complex plane.

Integrals in the complex plane of polynomial ratios are normally evaluated by factoring the denominator polynomial to determine the poles, and then summing the residues of the poles within the contour of integration. For the integrals (1) and (2), the residue method is highly laborious and essentially numeric in nature. However, advantage may be taken of the symmetry and assumptions in (1) and (2) to obtain an algebraic evaluation in matrix form using the coefficients of the polynomials of G .

II. Sampled Data Systems

The function G of Eq. (1) is given as

$$G(Z) = \frac{\sum_{i=0}^n b_{n-i} Z^i}{\sum_{i=0}^n a_{n-i} Z^i} \quad a_0 \neq 0 \quad (3)$$

The solution system of equations as derived in Ref. 1 is

$$\sum_{i=0}^n (a_{i-r} + a_{i+r}) M_i = B_r \quad r = 0, 1, \dots, n \quad (4)$$

$$M_0 = a_0 I_n$$

$$B_r = \begin{cases} \sum_{i=0}^n b_i^2 & r = 0 \\ 2 \sum_{i=0}^{n-r} b_i b_{i+r} & r = 1, 2, \dots, n \end{cases}$$

$$a_m = 0 \quad \begin{cases} m < 0 \\ m > n \end{cases}$$

Table 1 lists the matrix equations obtained from Eq. (4) for values of n from 1 to 4. The algebraic value of I_n may then be obtained using Cramer's rule (Ref. 2).

III. Continuous Systems

The function G of Eq. (2) is given as

$$G(S) = \frac{\sum_{i=0}^{n-1} b_{n-i} S^i}{\sum_{i=0}^n a_{n-i} S^i} \quad a_0 \neq 0 \quad (5)$$

The solution system of equations is derived in Appendix A using a method similar to the derivation of Eq. (4) in Ref. 1.

$$\begin{aligned} \sum_{i=0}^{n-1} a_{n-2r+i} M_i &= (-1)^r b_{n-r}^2 \\ &+ 2 \sum_{k=1}^{\infty} (-1)^{r+k} b_{n-r-k} b_{n-r+k} \end{aligned} \quad (6)$$

$$r = 0, 1, \dots, n-1$$

$$M_{n-1} = (-1)^{n-1} 2a_0 I_n$$

$$a_m = 0 \quad m < 0, m > n$$

$$b_m = 0 \quad m < 1, m > n$$

Table 2 lists the matrix equations obtained from Eq. (6) for values of n from 1 to 5. Similarly to the case of sampled data systems, I_n may be obtained through the use of Cramer's rule.

IV. Conclusion

Algebraic closed matrix forms have been presented for the evaluation of integrals (1) and (2). The closed forms provide the possibility of obtaining some insight into parameter sensitivity in addition to greatly reducing the computational complexity required by the normal method of evaluation by residues.

References

1. Jury, E. I., *Theory and Application of the Z-Transform Method*, Wiley, New York, 1964.
2. Apostol, T. M., *Calculus*, Vol. II, Wiley, New York, 1969.

Table 1. Matrix equation solutions to sampled-data systems integral

$n = 1$	$\begin{bmatrix} a_0 & a_1 \\ a_1 & a_0 \end{bmatrix} \begin{bmatrix} a_0 I_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} b_0^2 + b_1^2 \\ 2b_0 b_1 \end{bmatrix}$
$n = 2$	$\begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_0 + a_2 & a_1 \\ a_2 & 0 & a_0 \end{bmatrix} \begin{bmatrix} a_0 I_2 \\ M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} b_0^2 + b_1^2 + b_2^2 \\ 2(b_0 b_1 + b_1 b_2) \\ 2b_0 b_2 \end{bmatrix}$
$n = 3$	$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 + a_2 & a_1 + a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} a_0 I_3 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} b_0^2 + b_1^2 + b_2^2 + b_3^2 \\ 2(b_0 b_1 + b_1 b_2 + b_2 b_3) \\ 2(b_0 b_2 + b_1 b_3) \\ 2b_0 b_3 \end{bmatrix}$
$n = 4$	$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_0 + a_2 & a_1 + a_3 & a_2 + a_4 & a_3 \\ a_2 & a_3 & a_0 + a_4 & a_1 & a_2 \\ a_3 & a_4 & 0 & a_0 & a_1 \\ a_4 & 0 & 0 & 0 & a_0 \end{bmatrix} \begin{bmatrix} a_0 I_4 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} b_0^2 + b_1^2 + b_2^2 + b_3^2 + b_4^2 \\ 2(b_0 b_1 + b_1 b_2 + b_2 b_3 + b_3 b_4) \\ 2(b_0 b_2 + b_1 b_3 + b_2 b_4) \\ 2(b_0 b_3 + b_1 b_4) \\ 2b_0 b_4 \end{bmatrix}$

Table 2. Matrix equation solutions to continuous systems integral

$$n = 1 \quad [a_1] [2a_0 I_1] = [b_1^2]$$

$$n = 2 \quad \begin{bmatrix} a_2 & 0 \\ a_0 & a_1 \end{bmatrix} \begin{bmatrix} M_0 \\ -2a_0 I_2 \end{bmatrix} = \begin{bmatrix} b_2^2 \\ -b_1^2 \end{bmatrix}$$

$$n = 3 \quad \begin{bmatrix} a_3 & 0 & 0 \\ a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ 2a_0 I_3 \end{bmatrix} = \begin{bmatrix} b_3^2 \\ -b_2^2 + 2b_1 b_3 \\ b_1^2 \end{bmatrix}$$

$$n = 4 \quad \begin{bmatrix} a_4 & 0 & 0 & 0 \\ a_2 & a_3 & a_4 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_0 & a_1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ -2a_0 I_4 \end{bmatrix} = \begin{bmatrix} b_4^2 \\ -b_3^2 + 2b_2 b_4 \\ b_2^2 - 2b_1 b_3 \\ -b_1^2 \end{bmatrix}$$

$$n = 5 \quad \begin{bmatrix} a_5 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_5 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & a_0 & a_1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ 2a_0 I_5 \end{bmatrix} = \begin{bmatrix} b_5^2 \\ -b_4^2 + 2b_3 b_5 \\ b_3^2 - 2b_2 b_4 + 2b_1 b_5 \\ -b_2^2 + 2b_1 b_3 \\ b_1^2 \end{bmatrix}$$

Appendix A

Derivation of Eq. (6)

The integral to be evaluated is

$$I_n = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dS G(S)G(-S) \quad (\text{A-1})$$

where

$$G(S) = \frac{B(S)}{A(S)} = \frac{\sum_{i=0}^{n-1} b_{n-i} S^i}{\sum_{i=0}^n a_{n-i} S^i} \quad (\text{A-2})$$

Using a partial fraction expansion, assuming nonrepeated roots of $A(s)$

$$G(S)G(-S) = \frac{B(S)B(-S)}{A(S)A(-S)} = \sum_{k=1}^n \frac{R_k}{S + p_k} + \sum_{k=1}^n \frac{T_k}{S - p_k} \quad (\text{A-3})$$

where R_k is the residue of pole, $-p_k$, in the left-hand complex S -plane, and T_k is the residue of the symmetric pole, p_k , in the right-hand complex S -plane. From residue theory

$$I_n = \sum_{k=1}^n R_k \quad (\text{A-4})$$

The relationship of T_k to R_k is found as follows.

$$R_k = \left. \frac{B(S)B(-S)}{A'(S)A(-S)} \right|_{S=-p_k} = \frac{B(-p_k)B(p_k)}{A'(-p_k)A(p_k)} \quad (\text{A-5})$$

where $A'(S) \equiv d/dS A(S)$. Note that $d/dS A(-S) = -A'(-S)$. Therefore

$$T_k = - \left. \frac{B(S)B(-S)}{A(S)A'(-S)} \right|_{s=p_k} = - \frac{B(p_k)B(-p_k)}{A(p_k)A'(-p_k)} = -R_k \quad (\text{A-6})$$

From Eqs. (A-3) and (A-6)

$$B(S)B(-S) = \sum_{k=1}^n R_k \frac{A(S)A(-S)}{S + p_k} - \sum_{k=1}^n R_k \frac{A(S)A(-S)}{S - p_k} \quad (\text{A-7})$$

Eq. (6) is obtained by expanding both sides of Eq. (A-7) in polynomials of S and equating coefficients. Starting with the right side of Eq. (A-7), define $c_{i,k}$ such that

$$\frac{A(S)}{S + p_k} = \sum_{i=0}^{n-1} c_{i,k} S^i \quad (\text{A-8})$$

From Eqs. (A-2) and (A-8),

$$\sum_{i=0}^{n-1} c_{i,k} S^i = \frac{1}{S + p_k} \sum_{i=0}^n a_{n-i} S^i \quad (\text{A-9})$$

which becomes

$$\sum_{i=1}^n c_{i-1,k} S^i + p_k \sum_{i=0}^{n-1} c_{i,k} S^i = \sum_{i=0}^n a_{n-i} S^i \quad (\text{A-10})$$

For $i = n$

$$c_{n-1,k} = a_0 \quad (\text{A-11})$$

Using Eq. (A-8)

$$\frac{A(-S)}{S - p_k} = - \frac{A(-S)}{-S + p_k} = - \sum_{i=0}^{n-1} c_{i,k} (-S)^i \quad (\text{A-12})$$

From Eqs. (A-2) and (A-8)

$$\begin{aligned} \frac{A(S)A(-S)}{S + p_k} &= \sum_{i=0}^{n-1} c_{i,k} S^i \sum_{l=0}^n a_{n-l} (-S)^l \\ &= \left(\sum_{j=0}^{n-1} \sum_{i=0}^j + \sum_{j=n}^{2n-1} \sum_{i=j-n}^{n-1} \right) c_{i,k} a_{n-j+i} (-1)^{j-i} S^j \end{aligned} \quad (\text{A-13})$$

Similarly from Eqs. (A-2) and (A-12)

$$\begin{aligned} \frac{A(S)A(-S)}{S - p_k} &= - \sum_{i=0}^{n-1} c_{i,k} (-S)^i \sum_{l=0}^n a_{n-l} S^l \\ &= - \left(\sum_{j=0}^{n-1} \sum_{i=0}^j + \sum_{j=n}^{2n-1} \sum_{i=j-n}^{n-1} \right) c_{i,k} a_{n-j+i} (-1)^{i+j} S^j \end{aligned} \quad (\text{A-14})$$

Substituting Eqs. (A-13) and (A-14) into Eq. (A-7),

$$\begin{aligned}
B(S)B(-S) &= \sum_{k=1}^n R_k \left(\sum_{j=0}^{n-1} \sum_{i=0}^j + \sum_{j=n}^{2n-1} \sum_{i=j-n}^{n-1} \right) c_{i,k} a_{n-j+i} (-1)^{j-i} S^j \\
&\quad + \sum_{k=1}^n R_k \left(\sum_{j=0}^{m-1} \sum_{i=0}^j + \sum_{j=n}^{2n-1} \sum_{i=j-n}^{n-1} \right) c_{i,k} a_{n-j+i} (-1)^i S^j \\
&= \left(\sum_{j=0}^{n-1} \sum_{i=0}^j + \sum_{j=n}^{2n-1} \sum_{i=j-n}^{n-1} \right) S^j a_{n-j+i} (-1)^i (1 + (-1)^j) \sum_{k=1}^n R_k c_{i,k} \\
&= \left(\sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{i=0}^{2r} + \sum_{r=\lfloor \frac{n+1}{2} \rfloor}^{n-1} \sum_{i=2r-n}^{n-1} \right) S^{2r} a_{n-2r+i} (-1)^i 2 \sum_{k=1}^n R_k c_{i,k} \\
&= \sum_{r=0}^{n-1} S^{2r} \sum_{i=0}^{n-1} a_{n-2r+i} M_i
\end{aligned} \tag{A-15}$$

where $a_m = 0, m < 0, m > n$

$$M_i = (-1)^i 2 \sum_{k=1}^n R_k c_{i,k}$$

From Eqs. (A-4) and (A-11)

$$M_{n-1} = (-1)^{n-1} 2 \sum_{k=1}^n R_k c_{n-1,k} = (-1)^{n-1} 2 a_0 I_n \tag{A-16}$$

Using Eq. (A-2), the left side of Eq. (A-7) becomes

$$\begin{aligned}
B(S)B(-S) &= \sum_{i=0}^{n-1} b_{n-i} S^i \sum_{l=0}^{n-1} b_{n-l} (-S)^l \\
&= \left(\sum_{j=0}^{n-1} \sum_{i=0}^j + \sum_{j=n}^{2n-2} \sum_{i=j-n+1}^{n-1} \right) b_{n-i} b_{n-j+i} (-1)^{j-i} S^j \\
&= \sum_{r=0}^{n-1} S^{2r} \sum_{i=0}^{n-1} b_{n-i} b_{n-2r+i} (-1)^i
\end{aligned} \tag{A-17}$$

where $b_m = 0, m < 1, m > n$.

It is convenient to express the inner summation in Eq. (A-17) as

$$\sum_{i=0}^{n-1} b_{n-i} b_{n-2r+i} (-1)^i = b_{n-r}^2 (-1)^r + 2 \sum_{k=1}^{\infty} (-1)^{r+k} b_{n-r-k} b_{n-r+k} \quad (\text{A-18})$$

Therefore Eq. (A-17) becomes

$$B(S) B(-S) = \sum_{r=0}^{n-1} S^{2r} \left[b_{n-r}^2 (-1)^r + 2 \sum_{k=1}^{\infty} (-1)^{r+k} b_{n-r-k} b_{n-r+k} \right] \quad (\text{A-19})$$

where $b_m = 0, m < 1, m > n$

Equating the coefficients of S^{2r} in Eqs. (A-15) and (A-19) gives

$$\sum_{i=0}^{n-1} a_{n-2r+i} M_i = (-1)^r b_{n-r}^2 + 2 \sum_{k=1}^{\infty} (-1)^{r+k} b_{n-r-k} b_{n-r+k}$$

which is Eq. (6). It is conjectured that the results are also valid for repeated roots.